Integral Transforms

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Contents

0 Introduction

0.1 Preface

The module this course belongs to is assessed over two short, open-book examinations. As a consequence, these notes are very brief and contain very little proof or reasoning.

0.2 Useful Results from Complex Analysis

First, a few useful results from Complex Analysis.

Theorem (Residue Theorem)**.** Consider some complex function $f : \mathbb{C} \to \mathbb{C}$. Suppose f has a finite set of N poles $\{z_i\}$ inside some closed contour γ, then

$$\oint_\gamma f(z)\mathrm{d}z = 2\pi i \sum_{i=1}^{N} \mathrm{Res}\left[f(z), z_i\right]. \tag{1}$$

Theorem (Cauchy's Theorem)**.** If there are no singularities inside the closed contour γ, then

$$\oint_\gamma f(z)\mathrm{d}z = 0. \tag{2}$$

Theorem (Jordan's Lemma)**.** Consider some complex function $f : \mathbb{C} \to \mathbb{C}$ for which $f(z) \to 0$ as $|z| \to \infty$. Define γ_R^+ and γ_R^- to be semi-circular contours in the upper and lower half complex planes, respectively. In the limit $R \to \infty$,

$$\int_{\gamma_R^+} f(z)\mathrm{e}^{i\lambda z}\mathrm{d}z \to 0 \quad \text{if} \quad \lambda > 0 \tag{3}$$

$$\int_{\gamma_R^-} f(z)\mathrm{e}^{i\lambda z}\mathrm{d}z \to 0 \quad \text{if} \quad \lambda < 0 \tag{4}$$

0.3 Useful Integrals

For reference, here are some integrals that are useful to remember.

Example 1 (Gaussian Integral)**.** The integral of a Gaussian is given by

$$\int_{-\infty}^{\infty} \mathrm{e}^{-a(x-b)^2}\mathrm{d}x = \sqrt{\frac{\pi}{a}}, \tag{5}$$

for some constants $a, b \in \mathbb{C}$, $\Re(a) > 0$.

1 Fourier Transform

Before we can define the Fourier transform, we must first define the property of square integrability.

Definition (Square Integrable)**.** A function $f : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ is called square integrable if

$$\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x \tag{6}$$

is finite.

Now, let's define the Fourier transform,

Definition (Fourier Transform)**.** Consider a function $f : \mathbb{R} \to \mathbb{R} \quad \text{or} \quad \mathbb{C}$ which is square-integrable and decays to zero as $x \to \pm\infty$. Its Fourier transform is defined as the function $\tilde{f} : \mathbb{R} \to \mathbb{R} \quad \text{or} \quad \mathbb{C}$, where

$$\tilde{f}(k) = \mathcal{F}[f(x)](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-ikx} \mathrm{d}x. \tag{7}$$

The inverse Fourier transform of some function $\tilde{f}(k)$ is defined as

$$f(x) = \mathcal{F}^{-1}[\tilde{f}(k)](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) \mathrm{e}^{ikx} \mathrm{d}k. \tag{8}$$

Different conventions exist for the normalisation factors.

1.1 Properties and Useful Results

Let's go through some basic properties of the Fourier transform.

Property (Linearity)**.** For any square integrable functions $f(x)$ and $g(x)$ and scalars a and b, we have

$$\mathcal{F}[af(x) + bg(x)](k) = a\tilde{f}(k) + b\tilde{g}(k). \tag{9}$$

Property (Shift)**.** A shift in real space is a product with an exponential in k-space.

Suppose we shift $f(x)$ by $x_0 \in \mathbb{R}$. Its Fourier transform is

$$\mathcal{F}[f(x - x_0)] = \mathrm{e}^{-ikx_0} \tilde{f}(k). \tag{10}$$

Similarly, a shift in k-space is given by the product with an exponential in real space,

$$\mathcal{F}[\mathrm{e}^{ik'x} f(x)] = \tilde{f}(k - k'). \tag{11}$$

Property (Scaling)**.** Suppose we scale x by $\lambda \in \mathbb{R}$, the Fourier transform of $f(\lambda x)$ is given by

$$\mathcal{F}[f(\lambda x)](k) = \frac{1}{\lambda} \tilde{f}\left(\frac{k}{\lambda}\right) \tag{12}$$

Property (The Fourier Transform of a Real Function)**.** Consider a real function $f:\mathbb{R}\to\mathbb{R}$. Its Fourier transform $\tilde{f}(k)$ is not necessarily real. It can, however, be shown that

$$\tilde{f}(k)^* = \tilde{f}(-k). \tag{13}$$

Suppose that $f(x)$ is real and symmetric, then its Fourier transform is real. Conversely, if it is real and anti-symmetric, then its Fourier transform is imaginary.

Theorem (Fourier transform of a Gaussian)**.** Consider a Gaussian function

$$f(x) = \mathrm{e}^{-ax^2}, \tag{14}$$

its Fourier transform is given by another Gaussian,

$$\tilde{f}(k) = \frac{1}{\sqrt{2a}}\mathrm{e}^{-k^2/4a}. \tag{15}$$

Note that questions often expect you to start from the Fourier transform of $\exp(-x^2/2)$ and use the scaling theorem to find the FT for a different Gaussian.

Theorem (Derivative)**.** For any $n\in\mathbb{N}$, the Fourier transform of an n-differentiable, square-integrable function $f(x)$ is given by

$$\mathcal{F}\left[\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right](k) = (ik)^n\tilde{f}(k). \tag{16}$$

Analogically,

$$\mathcal{F}[x^n f(x)](k) = (i)^n\frac{\mathrm{d}^n\tilde{f}}{\mathrm{d}k^n}. \tag{17}$$

Theorem (Fourier transform in n dimensions)**.** The Fourier transform is easily extended to n dimensions,

$$\tilde{f}(\mathbf{k}) = \frac{1}{(2\pi)^{n/2}}\int \mathrm{d}^n r f(\mathbf{r})\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} \tag{18}$$

and, similarly,

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}}\int \mathrm{d}^n k f(\mathbf{r})\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} \tag{19}$$

1.1.1 The Dirac Delta Function

For reference, recall these basic properties of the Dirac delta,

$$\delta(-x) = \delta(x) \quad \text{and} \quad \delta(ax) = \frac{1}{|a|}\delta(x). \tag{20}$$

The delta function is differentiable. Denoting its nth-derivative $\delta^{(n)}(x)$, it can be shown that

$$\int_{-\infty}^{\infty}\delta^{(n)}(x-a)f(x)\mathrm{d}x = (-1)^n f^{(n)}(a). \tag{21}$$

The Dirac delta function is the derivative of the Heaviside function, i.e.

$$\Theta'(x) = \delta(x), \tag{22}$$

where the Heaviside step function is defined as

$$\Theta(x) = \begin{cases} 1 & \text{if} \quad x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

For some polynomial $g(x)$ with roots $\{x_1, \dots, x_N\}$, then it can be shown that

$$\delta(g(x)) = \sum_{i=1}^{N} \frac{\delta(x - x_i)}{|g'(x_i)|}. \tag{24}$$

In deriving the Fourier transform from the Fourier series, we are led to two important relations:

Theorem (Orthogonality Relation)**.**

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{ik(x-x')} \mathrm{d}k = \delta(x - x') \tag{25}$$

Theorem (Completeness Relation)**.**

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{ix(k-k')} \mathrm{d}x = \delta(k - k') \tag{26}$$

1.1.2 Convolution

Definition (Convolution)**.** The convolution between two functions f and g is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} \mathrm{d}y g(x - y) f(y). \tag{27}$$

It can be shown that convolution is symmetric and linear.

Theorem (Fourier Transform of a Convolution)**.** A convolution in real space is equal to multiplication in k-space.

$$\mathcal{F}[(f * g)(x)](k) = \sqrt{2\pi} \tilde{f}(k) \tilde{g}(k). \tag{28}$$

1.1.3 Parseval's Theorem

For some normed Hilbert space V, the normed inner product of two functions $f, g \in V$ is conserved under Fourier transform,

$$\int_{-\infty}^{\infty} \mathrm{d}x f^*(x) g(x) = \int_{-\infty}^{\infty} \mathrm{d}k \tilde{f}^*(k) \tilde{g}(k). \tag{29}$$

1.2 Applications of the Fourier Transform

We can apply the Fourier transform to solve a range of problems. In this course, we focus on solving differential equations and integral equations.

1.2.1 Differential Equations

Example 2 (Wave Equation)**.** In particular, we showed that the FT could be used to simplify the wave equation in x,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}, \tag{30}$$

where $v > 0$. Substituting for $y(x,t)$ its inverse FT, we were able to find a differential equation for $\tilde{y}(k,t)$ containing only derivatives in t,

$$\frac{\partial^2 \tilde{y}}{\partial t^2} = -v^2 k^2 \tilde{y}(k,t). \tag{31}$$

From this, we obtained a general solution for $y(x,t)$,

$$y(x,t) = \mathcal{F}^{-1}[\tilde{y}(k,t)](x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[A(k) \mathrm{e}^{ik(x-vt)} + B(k) \mathrm{e}^{ik(x+vt)} \right] \mathrm{d}k. \tag{32}$$

Example 3. We also considered the ODE

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - f = \delta(x). \tag{33}$$

Here, the Fourier transform of the delta function was used to obtain the particular solution,

$$f_{\mathrm{p}}(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ikx}}{k^2+1} \mathrm{d}k = -\frac{1}{2} \mathrm{e}^{-|x|}, \tag{34}$$

where Jordan's lemma and Cauchy's theorem were applied to solve the integral.

1.2.2 Integral Equations

The convolution theorem makes the Fourier transform a powerful tool in solving integral equations involving convolutions.

Example 4. Consider the integral

$$h(x) = \mathrm{e}^{\mathrm{i3x}} + \int_{-\infty}^{\infty} h(x-y) \mathrm{e}^{-|y|} \mathrm{d}y. \tag{35}$$

Noticing the second term on the RHS is a convolution, we can take the Fourier transform to find an equation for $\tilde{h}(k)$,

$$\tilde{h}(k) = \sqrt{2\pi} \frac{k^2+1}{k^2-1} \delta(k-3). \tag{36}$$

Applying the inverse FT, we find

$$h(x) = \frac{5}{4} \mathrm{e}^{i3x}. \tag{37}$$

1.3 The Discrete Fourier Transform

We consider two cases of the discrete Fourier transform (DFT).

1.3.1 Continuous $x \in [0, L)$

Consider some function $f(x)$, where $x \in [0, L)$. We can expand $f(x)$,

$$f(x) = \frac{1}{\sqrt{L}} \sum_k \tilde{f}(k) \mathrm{e}^{ikx}, \tag{38}$$

$$\tilde{f}(k) = \frac{1}{\sqrt{L}} \int_0^L \mathrm{d}x f(x) \mathrm{e}^{-ikx}. \tag{39}$$

Here, k is discretised into

$$k = \tilde{n} \frac{2\pi}{L}, \tag{40}$$

where

$$\tilde{n} = 0, \pm 1, \pm 2, \ldots. \tag{41}$$

1.3.2 Discrete $x \in [0, L)$

Say x is now discrete, such that

$$x = n \Delta x, \tag{42}$$

where

$$n = 0, 1, 2, \ldots, N-1, \tag{43}$$

and $N = L/\Delta x$. The function $f(x)$ is discretised into f_n. The discrete Fourier transform of f_n is

$$f_n = \frac{1}{\sqrt{N\Delta x}} \sum_{\tilde{n}=0}^{N-1} \tilde{f}_{\tilde{n}} \mathrm{e}^{i2\pi\tilde{n}n/N}, \tag{44}$$

$$\tilde{f}_{\tilde{n}} = \frac{1}{\sqrt{N\Delta x}} \sum_{n=0}^{N-1} f_n \mathrm{e}^{-i2\pi\tilde{n}n/N}. \tag{45}$$

This can be written in matrix form,

$$\tilde{\mathbf{f}} = M\mathbf{f} \tag{46}$$

where $\mathbf{f}$ and $\tilde{\mathbf{f}}$ are column vectors with elements f_n and $\tilde{f}_{\tilde{n}}$, respectively. The matrix elements are given by

$$M_{ij} = \frac{1}{\sqrt{N\Delta x}} \mathrm{e}^{-2\pi ij/N}. \tag{47}$$

Whilst the computation of $\tilde{\mathbf{f}}$ scales with $\mathcal{O}(N^2)$, the symmetries in M allow for computation in $\mathcal{O}(N \log N)$ with the Fast Fourier Transform algorithm.

2 Laplace Transform

Whilst the Fourier transform is a powerful tool, it has two key restrictions. It is not defined for functions which do not decay to zero and there is no way of incorporating initial or boundary conditions – we can only obtain a particular integral. The Laplace transform gets around these restrictions but introduces a new one.

Definition (Laplace Transform)**.** Consider a function $f(t) : \mathbb{R} \to \mathbb{C}$. The Laplace transform is only defined for $f(t)$ if

$$f(t) = \Theta(t)f(t) = \begin{cases} f(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases} \tag{48}$$

Hereafter, we will omit the Heaviside function and assume all functions we are interested in vanish for $t < 0$. For such a function $f(t)$, the Laplace transform is defined as

$$\hat{f}(s) = \mathcal{L}[f(t)](s) = \int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t, \tag{49}$$

where the domain of s is restricted such that for all s $f(t)\exp(-st) \to 0$ for $t \to \infty$, i.e. the Laplace transform exists for functions that grow no more than exponentially fast.

Definition (Inverse Laplace Transform and the Browmwich inversion formula)**.** We also have the inverse Laplace transform,

$$f(t) = \mathcal{L}^{-1}[\hat{(f)}(s)](t). \tag{50}$$

Sometimes, the inverse LT can be found by recognition. When this isn't possible, we use the inversion formula,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}(s)\mathrm{e}^{st}\mathrm{d}s, \tag{51}$$

where c is a real constant such that the Browmwich inversion contour $\{c - i\infty \to c + i\infty\}$ lies to the right of all singularities of$\hat{f}(s)$. This requirement ensures that $f(t) = 0$ for all $t < 0$.

Proposition. Suppose that $\hat{f}(s)$ has a finite number of isolated singularities $\{s_1, \ldots, s_N\}$ and that $\hat{f}(s) \to 0$ as $|s| \to \infty$, then the inverse Laplace transform is given by

$$f(t) = \sum_{n=1}^{N} \mathrm{Res}[\hat{f}(s)\mathrm{e}^{st}, s_n] \tag{52}$$

for $t \geq 0$ and vanishes for $t < 0$.

2.1 Properties of the Laplace Transform

For any scalars $a, b \in \mathbb{C}$ and functions $f, g : \mathbb{R}^+ \to \mathbb{C}$ that vanish for $t < 0$, we have the following properties:

Property (Linearity)**.**

$$\mathcal{L}[af + bg](s) = a\mathcal{L}[f](s) + b\mathcal{L}[g](s), \tag{53}$$

Property (Translation)**.** Consider a time shift by $t_0 > 0$,

$$\mathcal{L}[f(t-t_0)H(t-t_0)](s) = \mathrm{e}^{-st_0}\hat{f}(s). \tag{54}$$

Similarly, consider a frequency shift by s_0,

$$\mathcal{L}[e^{s_0 t}f(t)](s) = \hat{f}(s-s_0). \tag{55}$$

Property (Scaling)**.** Consider time scaling by a factor of $\lambda > 0$,

$$\mathcal{L}[f(\lambda t)](s) = \frac{1}{\lambda}\hat{f}\left(\frac{s}{\lambda}\right), \tag{56}$$

where we impose the constraint on λ such that $f(\lambda t)$ vanishes for $t < 0$.

Property (Transform of a Derivative)**.** The Laplace transform of a first derivative is

$$\mathcal{L}[f'(t)](s) = s\hat{f}(s) - f(0). \tag{57}$$

Repeating this process n times, assuming f to be n-times differentiable and with an nth derivative of exponential type,

$$\mathcal{L}[f^{(n)}(t)](s) = s^n\hat{f}(s) - \sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0). \tag{58}$$

Property (Derivative of a Transform)**.** The nth derivative of the Laplace transform is given by

$$\mathcal{L}[t^n f(t)](s) = (-1)^n\frac{\mathrm{d}^n\hat{f}}{\mathrm{d}s^n}. \tag{59}$$

From this property, we can easily calculate the Laplace transform of many functions, such as

$$\mathcal{L}[t^n](s) = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}s^n}\frac{1}{s} = \frac{n!}{s^{n+1}} \tag{60}$$

Property (Taylor Series)**.** Suppose that $f(t)$ can be written in a power series as

$$f(t) = \sum_{n=0}^{\infty}\frac{a_n}{n!}t^n, \tag{61}$$

then, following from our result for t^n above, its Laplace transform can be written

$$\hat{f}(s) = \sum_{n=0}^{\infty}\frac{a_n}{s^{n+1}}. \tag{62}$$

Property (Asymptotic Limits)**.**

$$s\hat{f}(s) \to \begin{cases} f(0) & \text{as } s\to\infty \\ f(\infty) & \text{as } s\to 0, \end{cases} \tag{63}$$

where the second case requires the limit of f at ∞ to exist. These cases are often referred to as the initial and final value theorems, respectively.

Property (Integrals)**.** The Laplace transform of an integral is, for $0 \leq a \leq t$,

$$\mathcal{L}\left[\int_a^t f(t')\mathrm{d}t'\right](s) = \frac{1}{s}\hat{f}(s) - \frac{1}{s}\int_0^a f(t')\mathrm{d}t', \tag{64}$$

where we take the integral on the RHS as 0 when $a = 0$.

Property (Periodic Functions)**.** Consider a function $f(t)$ periodic with period T, its Laplace transform satisfies

$$\hat{f}(s) = \frac{1}{1 - \mathrm{e}^{-sT}}\int_0^T \mathrm{d}t f(t)\mathrm{e}^{-st}. \tag{65}$$

Theorem (Convolution Theorem)**.** Since f and g vanish for negative t, the definition of the convolution from before simplifies to

$$(f * g)(t) = \int_0^t g(t - t')f(t')\mathrm{d}t' \tag{66}$$

Just like for the Fourier transform, the Laplace transform of a convolution is given by

$$\mathcal{L}[(f * g)(t)](s) = \hat{f}(s)\hat{g}(s). \tag{67}$$

2.2 Applications of the Laplace Transform

The Laplace transform converts ODEs to algebraic equations and PDEs to ODEs. It is particularly useful in solving DEs when the initial conditions at $t = 0$ are known.

The Laplace transform can also be used to solve integral equations, transforming a difficult integral in time-space into one of a more favourable form in frequency space. The solution is found by solving the frequency integral and applying the inverse Laplace transform.

3 Summary

3.1 Common Fourier Transforms

$$\mathcal{F}[1](k) = \sqrt{2\pi}\delta(k) \tag{68}$$
$$\mathcal{F}[\mathrm{e}^{iax}](k) = \sqrt{2\pi}\delta(k-a) \tag{69}$$
$$\mathcal{F}[\delta(x-a)](k) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-ika} \tag{70}$$
$$\mathcal{F}[\cos(ax)](k) = \sqrt{2\pi}\frac{1}{2}\left(\delta(k-a)+\delta(k+a)\right) \tag{71}$$
$$\mathcal{F}[\sin(ax)](k) = \sqrt{2\pi}\frac{1}{2i}\left(\delta(k-a)-\delta(k+a)\right) \tag{72}$$
$$\mathcal{F}[x^n](k) = i^n\sqrt{2\pi}\delta^{(n)}(k) \quad \text{for } n \in \mathbb{N} \tag{73}$$
$$\mathcal{F}[\cos(ax^2)](k) = \frac{1}{\sqrt{2a}}\cos\left(\frac{k^2}{4a}-\frac{\pi}{4}\right) \tag{74}$$
$$\mathcal{F}[\sin(ax^2)](k) = \frac{-1}{\sqrt{2a}}\sin\left(\frac{k^2}{4a}-\frac{\pi}{4}\right) \tag{75}$$
$$\mathcal{F}\left[\frac{1}{x}\right](k) = -i\sqrt{\frac{\pi}{2}}\mathrm{sgn}(k) \tag{76}$$

3.2 Properties of the Fourier Transform

$$\mathcal{F}[af(x)+bg(x)](k) = a\tilde{f}(k)+b\tilde{g}(k) \tag{77}$$
$$\mathcal{F}[f(x-x_0)] = \mathrm{e}^{-ikx_0}\tilde{f}(k) \tag{78}$$
$$\mathcal{F}[\mathrm{e}^{ik'x}f(x)] = \tilde{f}(k-k') \tag{79}$$
$$\mathcal{F}[f(\lambda x)](k) = \frac{1}{\lambda}\tilde{f}\left(\frac{k}{\lambda}\right) \tag{80}$$
$$\mathcal{F}\left[\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right](k) = (ik)^n\tilde{f}(k) \tag{81}$$
$$\mathcal{F}[x^n f(x)](k) = (i)^n\frac{\mathrm{d}^n\tilde{f}}{\mathrm{d}k^n} \tag{82}$$
$$\mathcal{F}[(f*g)(x)](k) = \sqrt{2\pi}\tilde{f}(k)\tilde{g}(k) \tag{83}$$

3.3 Common Laplace Transforms

Assume the region of convergence is $\mathrm{Re}(s) > 0$ unless stated otherwise.

$$\mathcal{L}[1] = \frac{1}{s} \tag{84}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad \text{for } n \geq 0 \tag{85}$$

$$\mathcal{L}[\mathrm{e}^{\lambda t}] = \frac{1}{s-\lambda}, \quad \text{for } \mathrm{Re}(s) > \mathrm{Re}(\lambda) \tag{86}$$

$$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2} \tag{87}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \tag{88}$$

$$\mathcal{L}[\sinh(\omega t)] = \frac{\omega}{s^2-\omega^2}, \quad \text{for } \mathrm{Re}(s) > |\omega| \tag{89}$$

$$\mathcal{L}[\cosh(\omega t)] = \frac{s}{s^2-\omega^2}, \quad \text{for } \mathrm{Re}(s) > |\omega| \tag{90}$$

$$\mathcal{L}[\delta(t-t')] = \mathrm{e}^{-st'} \tag{91}$$

3.4 Properties of the Laplace Transform

$$\mathcal{L}[af+bg](s) = a\mathcal{L}[f](s) + b\mathcal{L}[g](s) \tag{92}$$

$$\mathcal{L}[f(t-t_0)H(t-t_0)](s) = \mathrm{e}^{-st_0}\hat{f}(s) \tag{93}$$

$$\mathcal{L}[f(\lambda t)](s) = \frac{1}{\lambda}\hat{f}\left(\frac{s}{\lambda}\right) \tag{94}$$

$$\mathcal{L}[f^{(n)}(t)](s) = s^n\hat{f}(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0) \tag{95}$$

$$\mathcal{L}[t^n f(t)](s) = (-1)^n \frac{\mathrm{d}^n \hat{f}}{\mathrm{d}s^n} \tag{96}$$

$$\mathcal{L}\left[\int_a^t f(t')\mathrm{d}t'\right](s) = \frac{1}{s}\hat{f}(s) - \frac{1}{s}\int_0^a f(t')\mathrm{d}t' \tag{97}$$

$$\mathcal{L}[(f*g)(t)](s) = \hat{f}(s)\hat{g}(s) \tag{98}$$